Section 15.2:
Double Integrals Over General Regions

What We'll Learn In Section 15.2

1. Double Integrals Over General Regions

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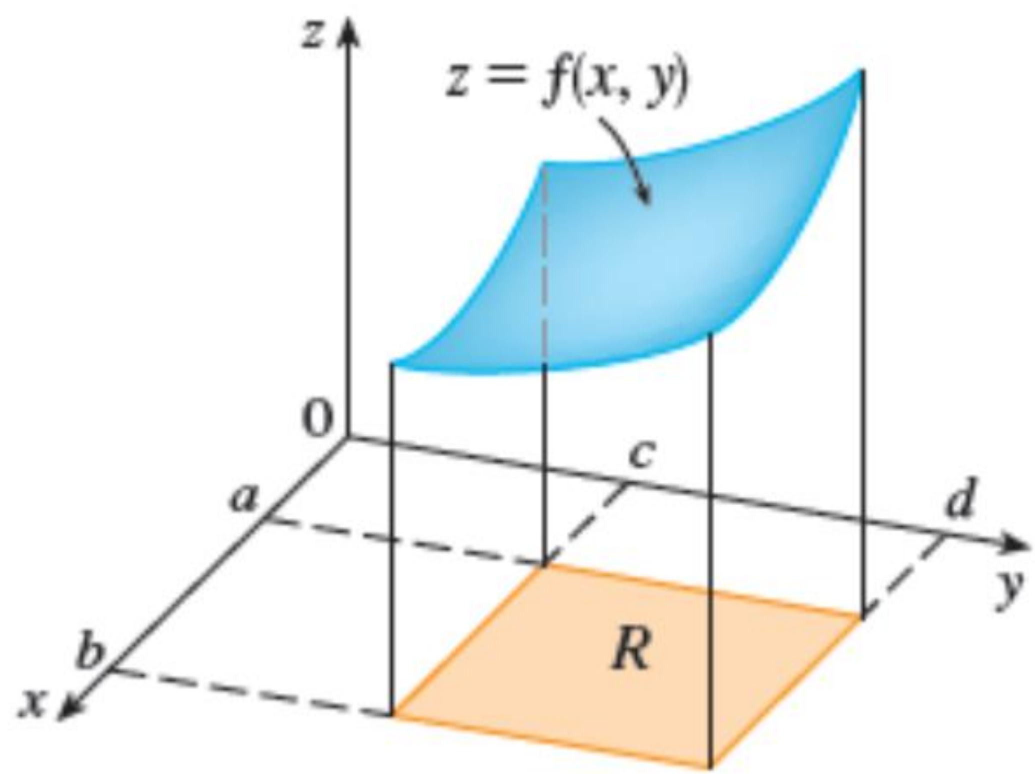
In section 15.1, we learned how to integrate a 2-variable function $f(x,y)$ over a rectangle $R = \{ (x,y) \mid a \le x \le b\, , c \le y \le d \}$.

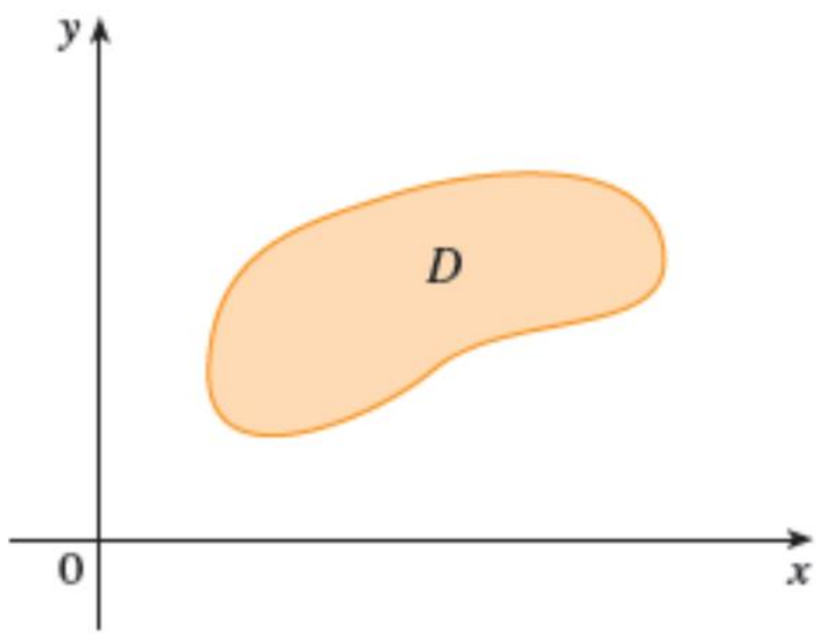
$$\iint_R f(x,y)\, dA$$

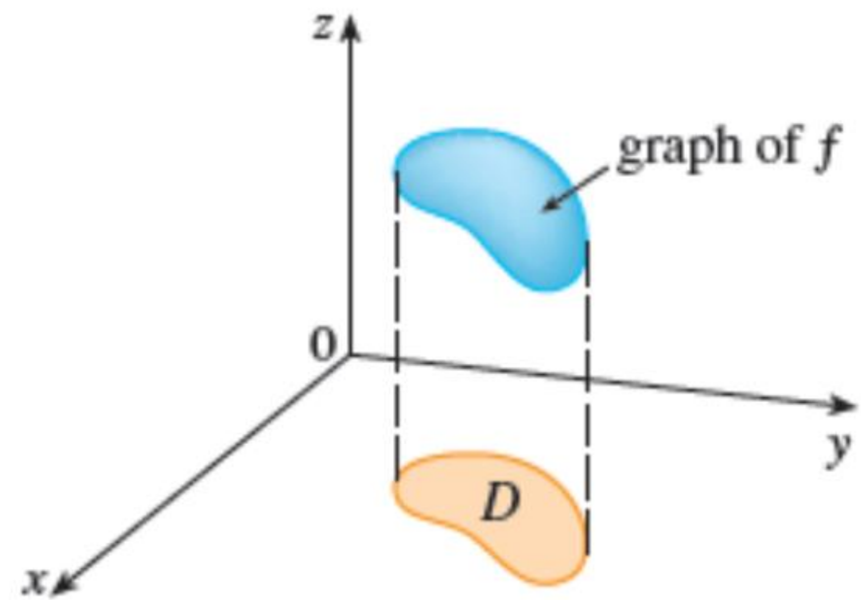


1. Double Integrals Over General Regions

Now, we want to integrate a 2-variable function $f(x, y)$ over a general (compact) region D.

$$\iint_D f(x, y)\, dA$$



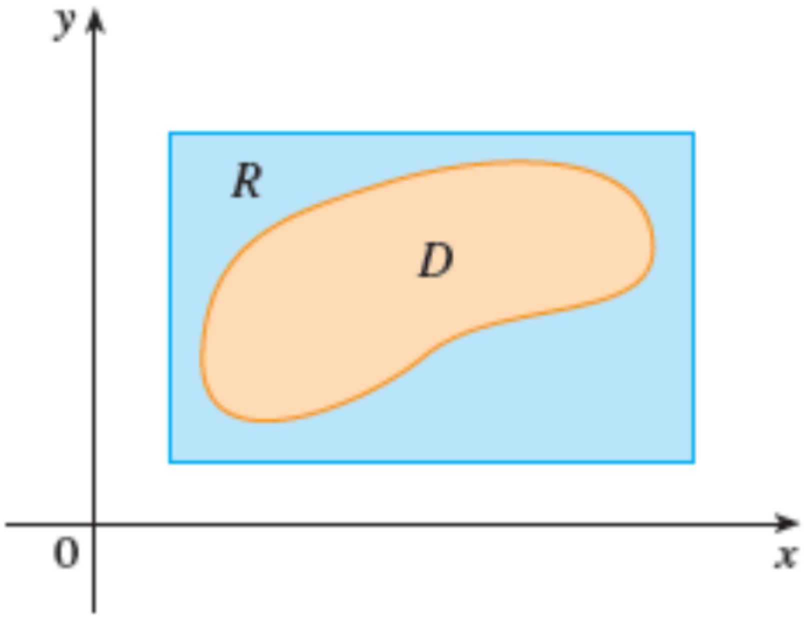


1. Double Integrals Over General Regions

Now, we want to integrate a 2-variable function $f(x, y)$ over a general (compact) region D.

How?

1. Draw a rectangle around D



1. Double Integrals Over General Regions

Now, we want to integrate a 2-variable function $f(x, y)$ over a general (compact) region D.

How?

2. Extend the definition of f to the entire rectangle by defining another 2-variable function $F(x, y)$ by…

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

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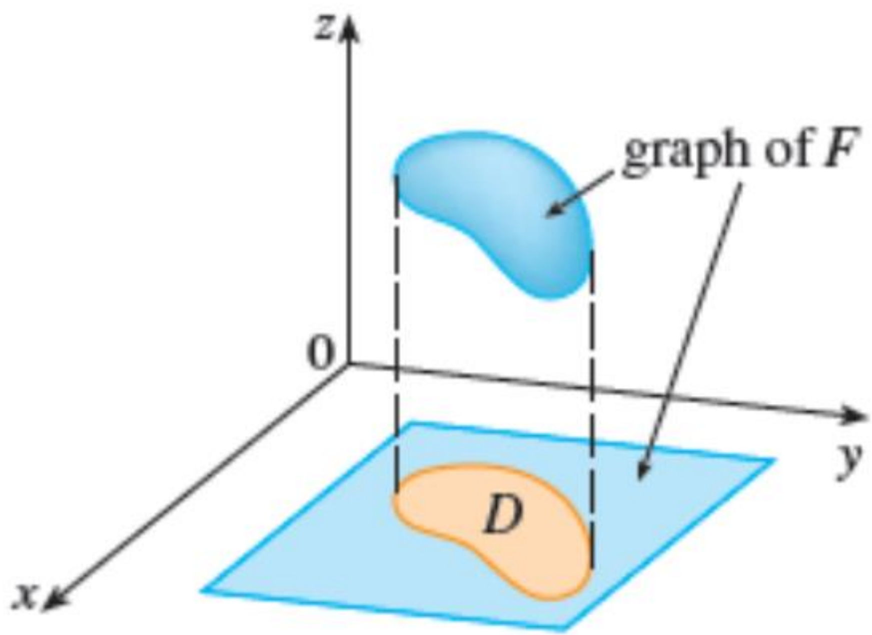


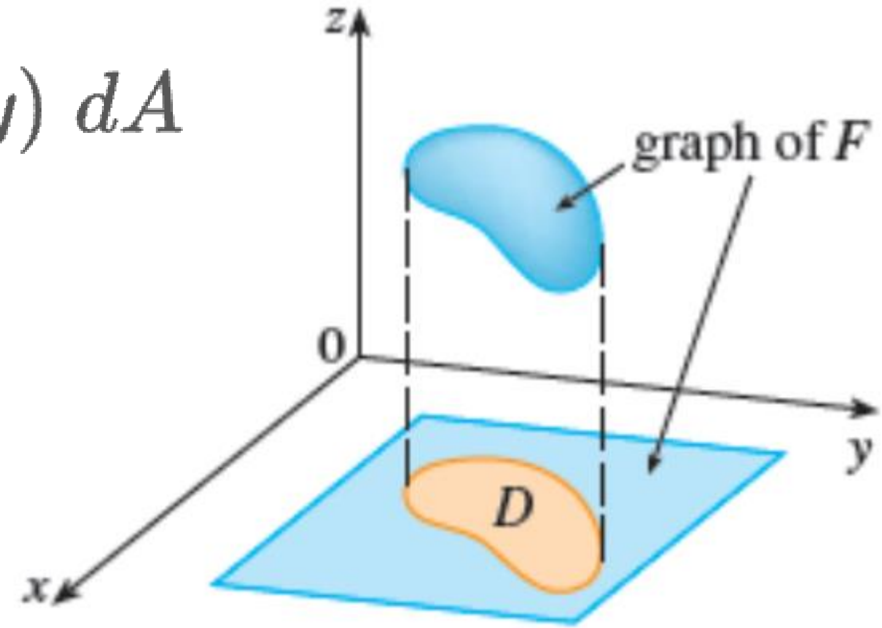
1. Double Integrals Over General Regions

Now, we want to integrate a 2-variable function $f(x, y)$ over a general (compact) region D.

How?

3. Then define the integral of f by…

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$



1. Double Integrals Over General Regions

We will learn how to integrate over 3 special types of regions in this section…

1) Type I regions
2) Type II regions
3) Regions that are finite disjoint unions of type I and type II regions (disjoint except for the boundaries)

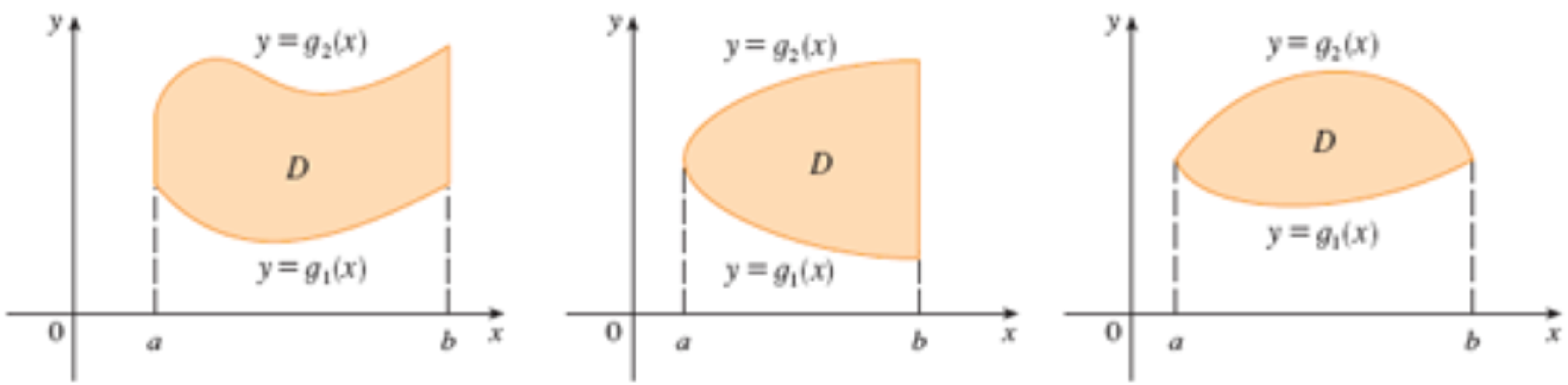
1. Double Integrals Over General Regions

Type I regions

A type I region is a region bounded by 2 functions of x from $x = a$ to $x = b$.

$$D = \{\, (x, y) \mid a \le x \le b\,, g_1(x) \le y \le g_2(x) \,\}$$

Some type I regions



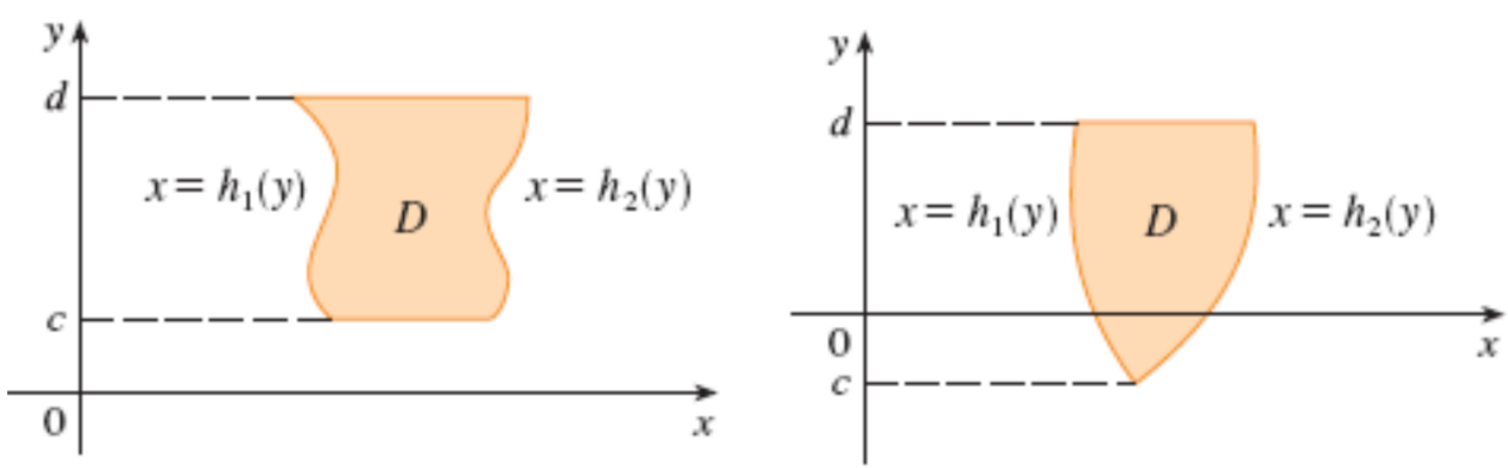
1. Double Integrals Over General Regions

Type II regions

A type II region is a region bounded by 2 functions of y from $y = c$ to $y = d$.

$$D = \{ (x, y) \mid c \leq y \leq d \,, h_1(y) \leq x \leq h_2(y) \}$$

Some type II regions



1. Double Integrals Over General Regions

Double Integral Over a Type I Or Type II Region

To calculate a double integral of 2-variable function $f(x, y)$ over a type I or type II region, set it up as an iterated integral by…

Type I: $$\iint\limits_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Type II: $$\iint\limits_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Explain/why?

1. Double Integrals Over General Regions

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1. Double Integrals Over General Regions

<u>Ex 1:</u>

Evaluate $\iint\limits_D x + 2y \, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

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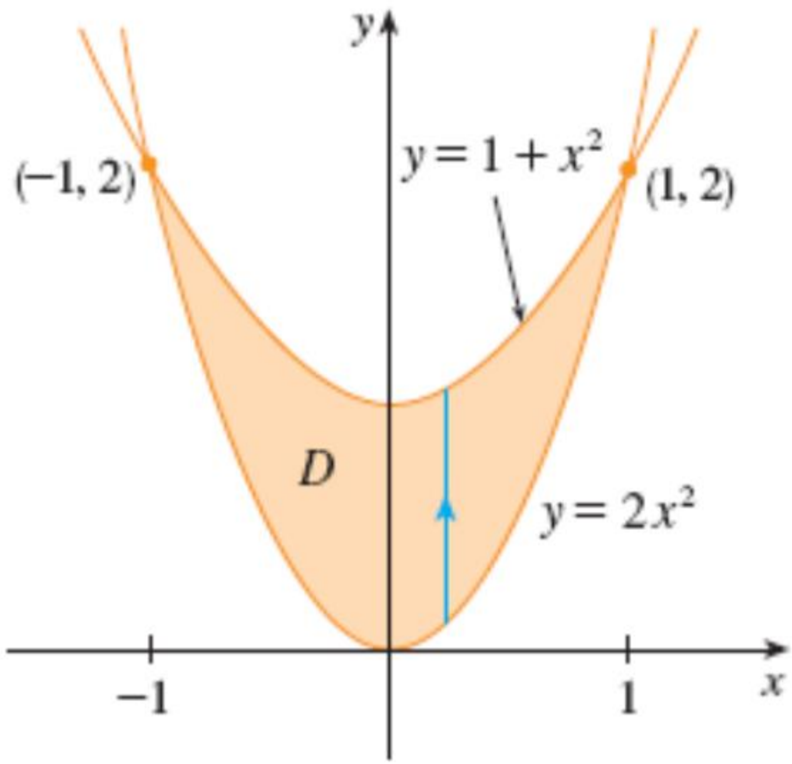

1. Double Integrals Over General Regions

Ex 2:

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region *D* in the *xy*-plane bounded by the line $y = 2x$ and the parabola $y = x^2$. (do twice)

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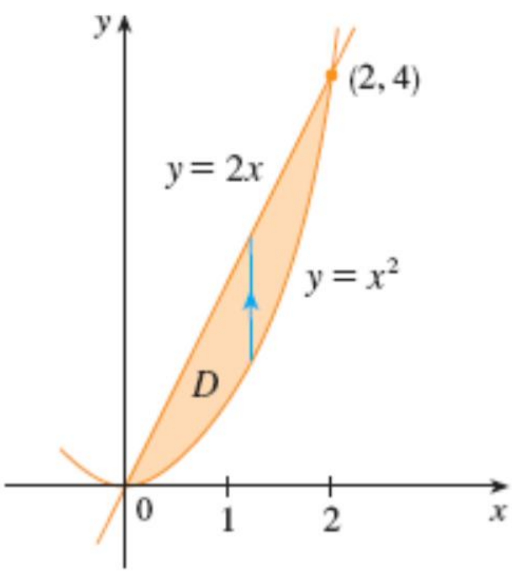

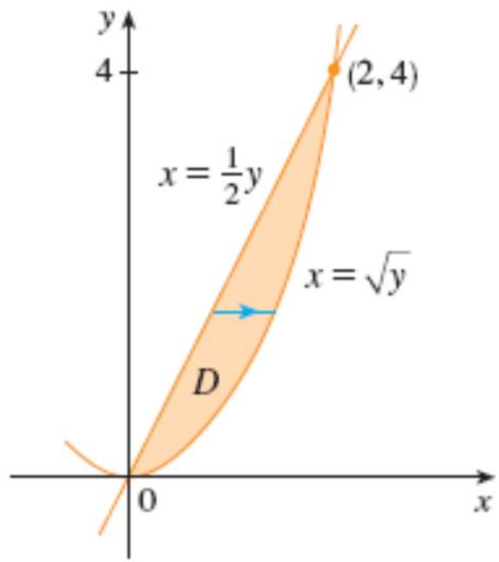

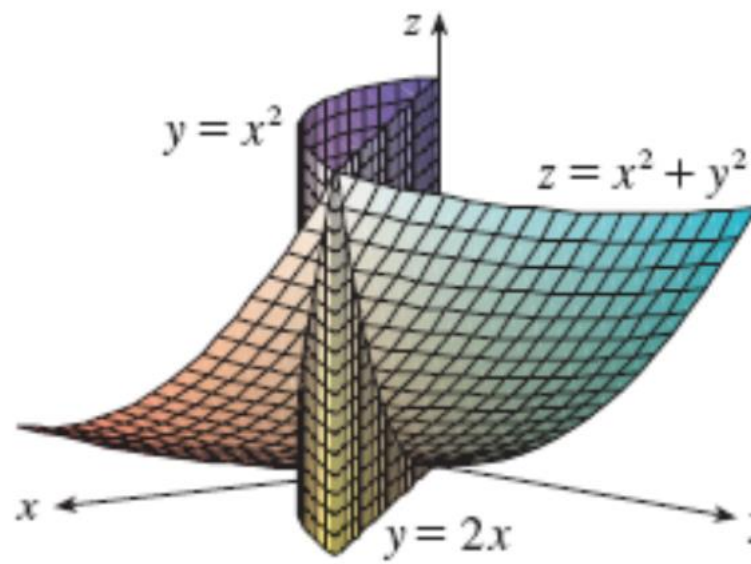

1. Double Integrals Over General Regions

<u>Ex 3:</u>

Evaluate $\iint\limits_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. (do only as a type II region for now)

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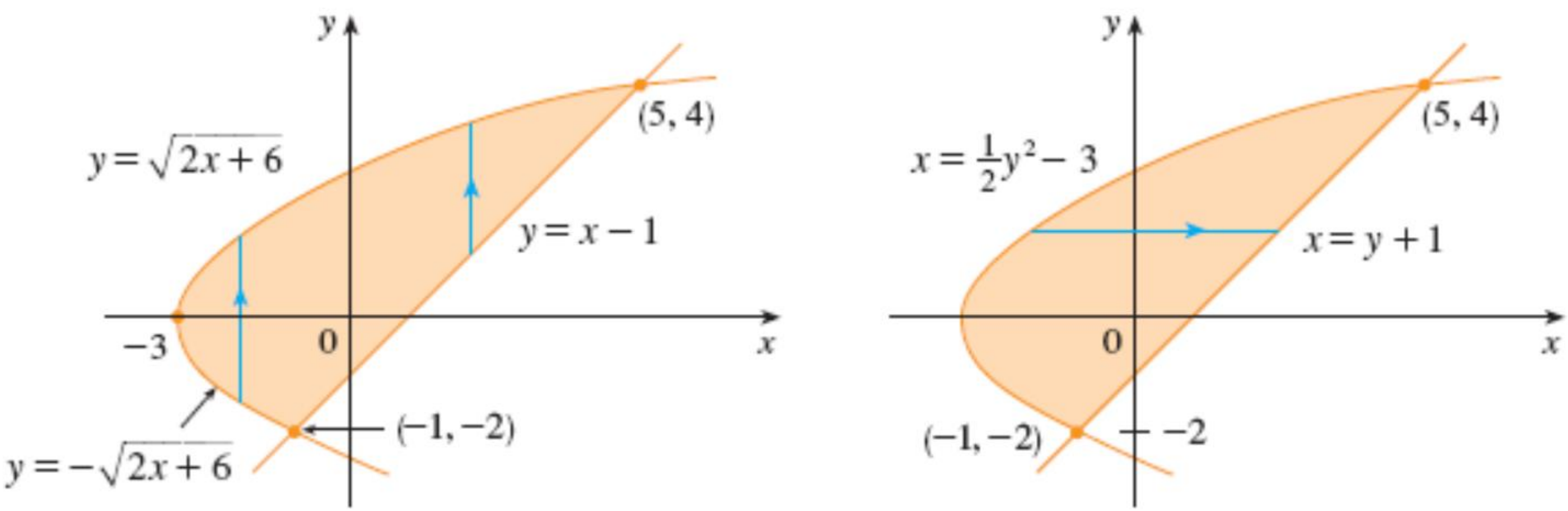

(a) D as a type I region (b) D as a type II region

1. Double Integrals Over General Regions

Ex 4:

Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

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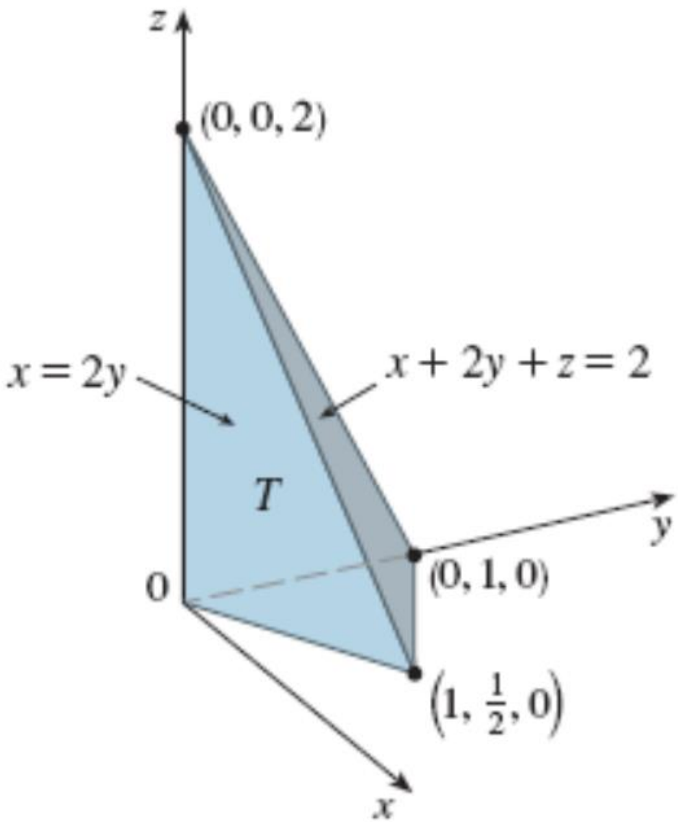

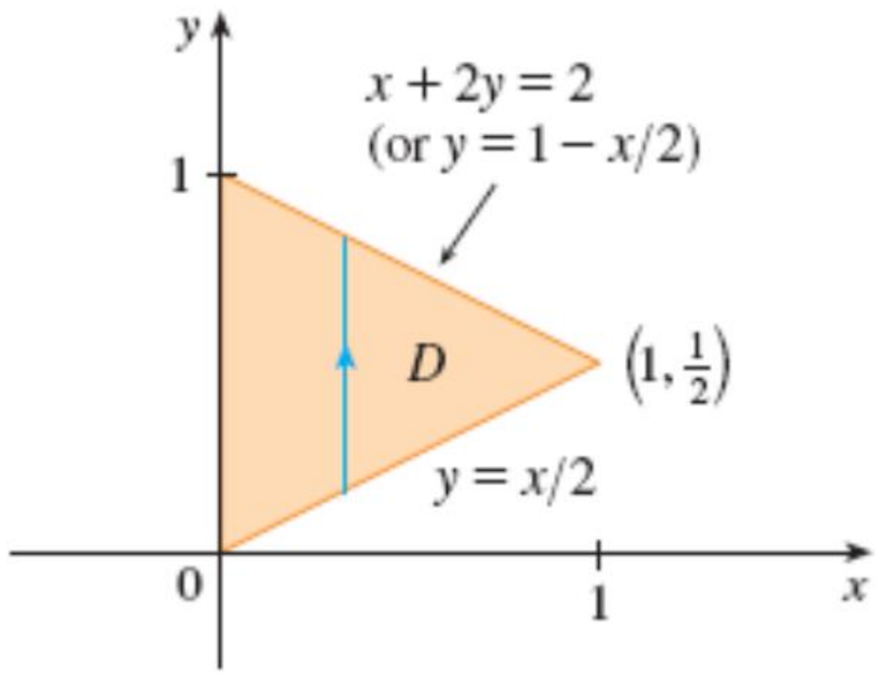

1. Double Integrals Over General Regions

<u>Ex 5:</u>

Evaluate the iterated integral $\displaystyle\int_0^1 \int_x^1 \sin(y^2)\ dydx$

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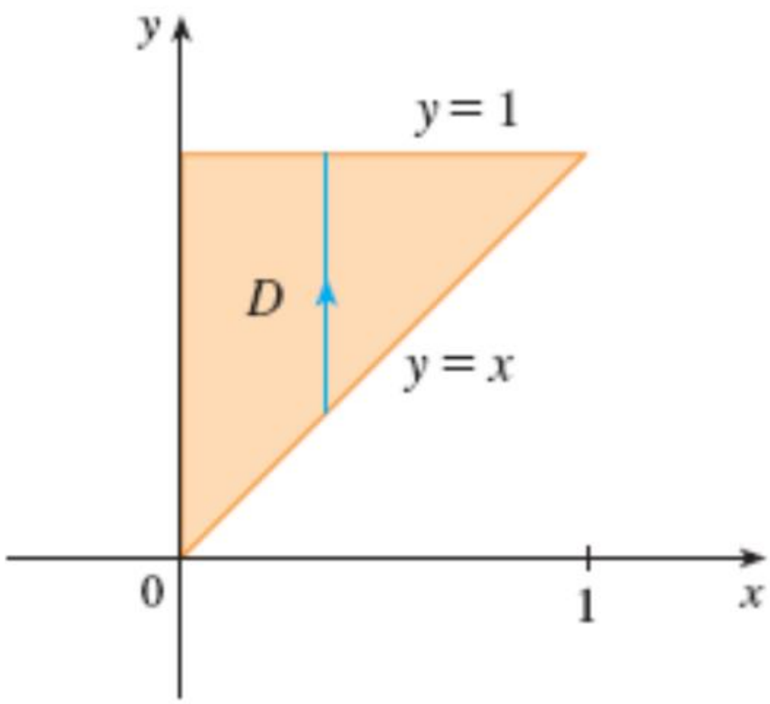
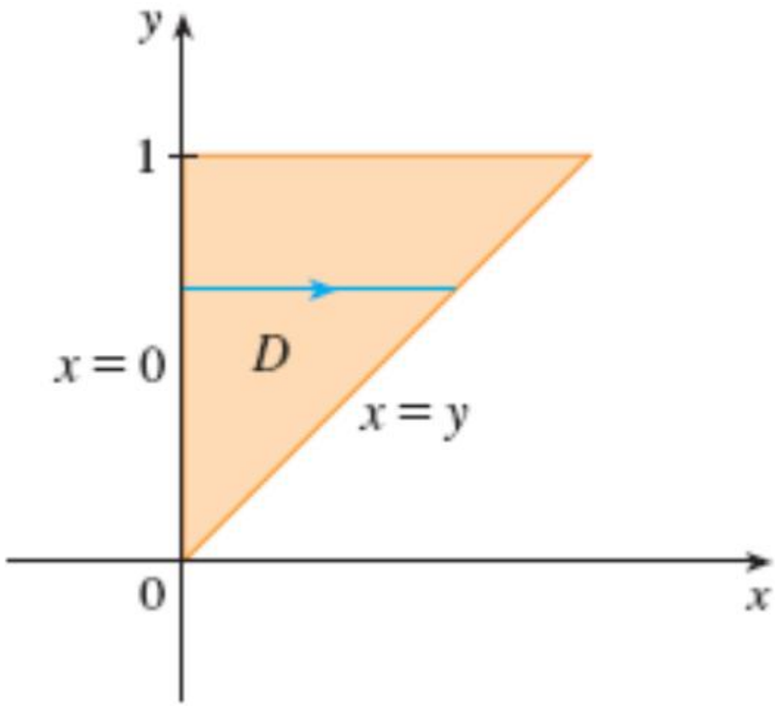




1. Double Integrals Over General Regions

Some properties of double integrals

$$\boxed{6} \quad \iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

$$\boxed{7} \quad \iint_D cf(x,y)\, dA = c \iint_D f(x,y)\, dA \qquad \text{where } c \text{ is a constant}$$

$\boxed{8}$ If $f(x,y) \geqslant g(x,y)$ for all (x,y) in D, then

$$\iint_D f(x,y)\, dA \geqslant \iint_D g(x,y)\, dA$$

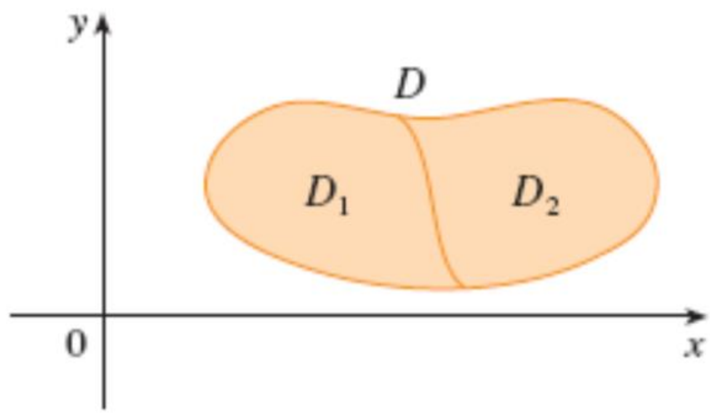
1. Double Integrals Over General Regions

Some properties of double integrals

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries, then

9

$$\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA$$



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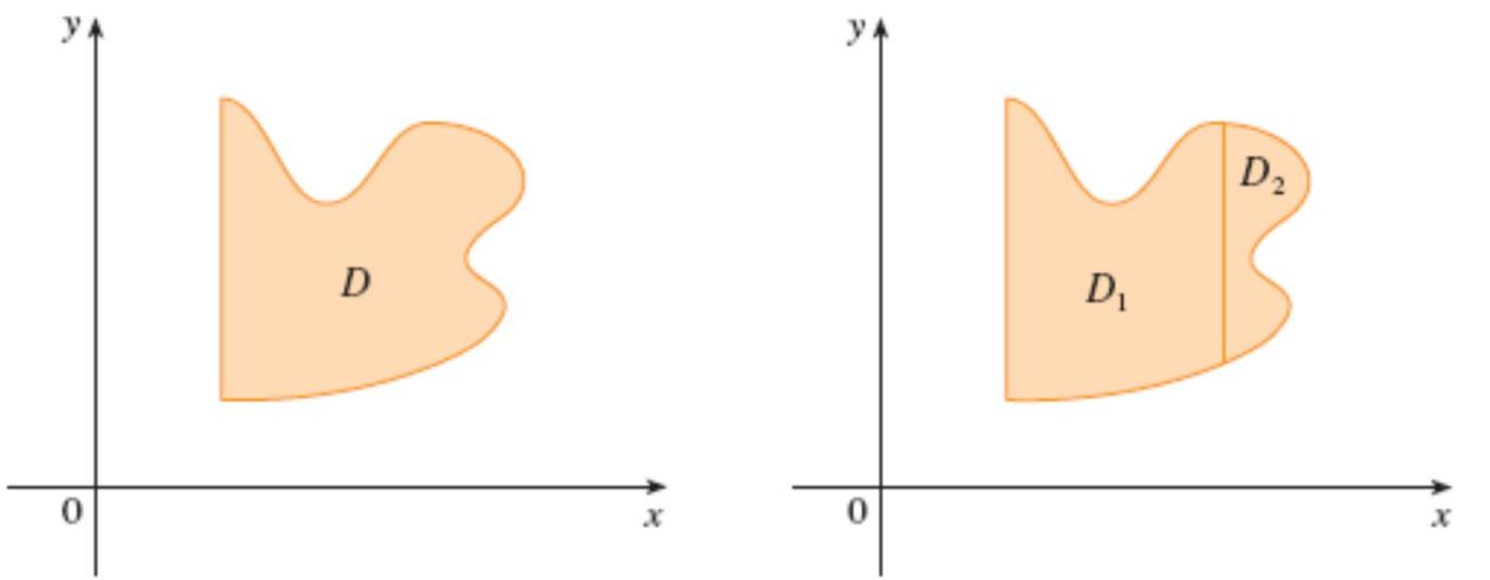
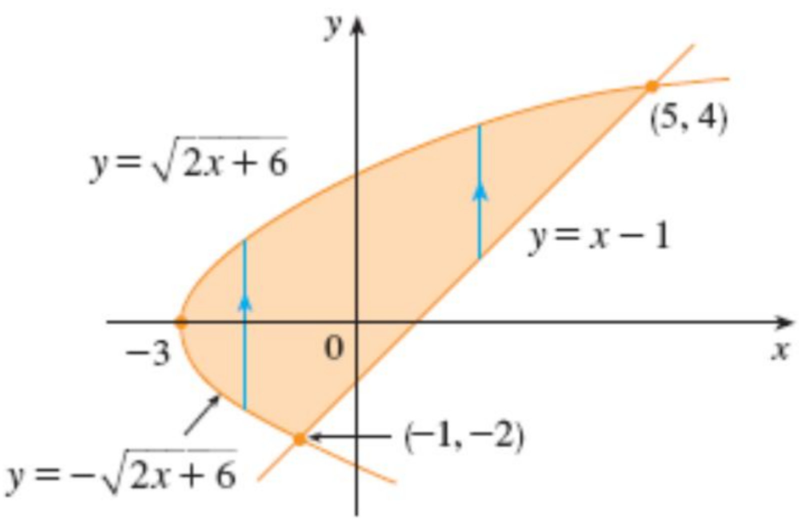


(a) D is neither type I nor type II.

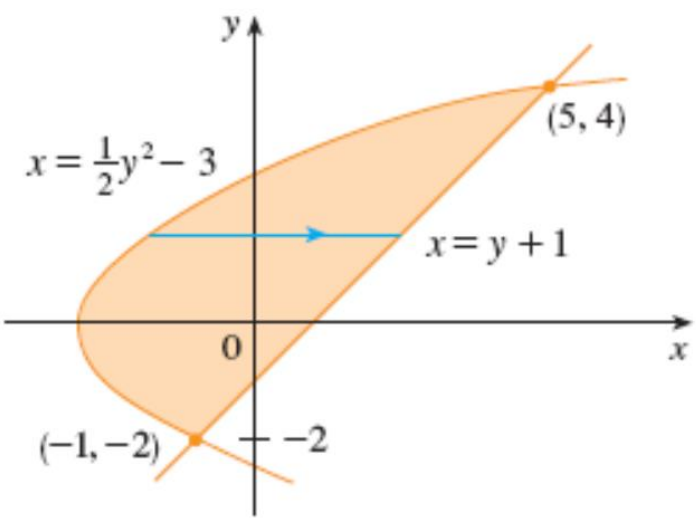
(b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

1. Double Integrals Over General Regions

<u>Ex 3 again:</u>
Evaluate $\iint\limits_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. (do as a type I region)



(a) D as a type I region



(b) D as a type II region

1. Double Integrals Over General Regions

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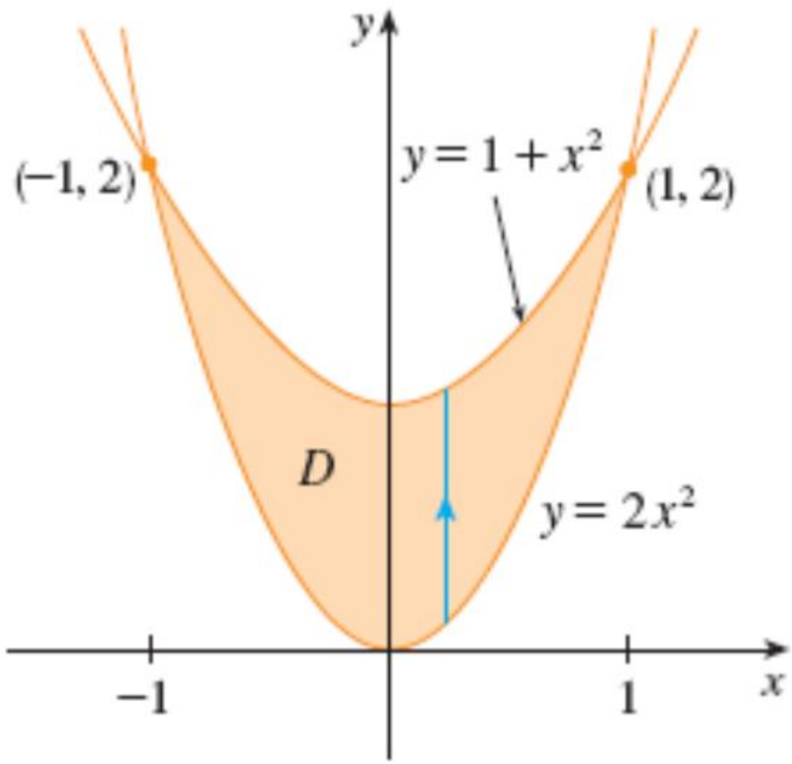

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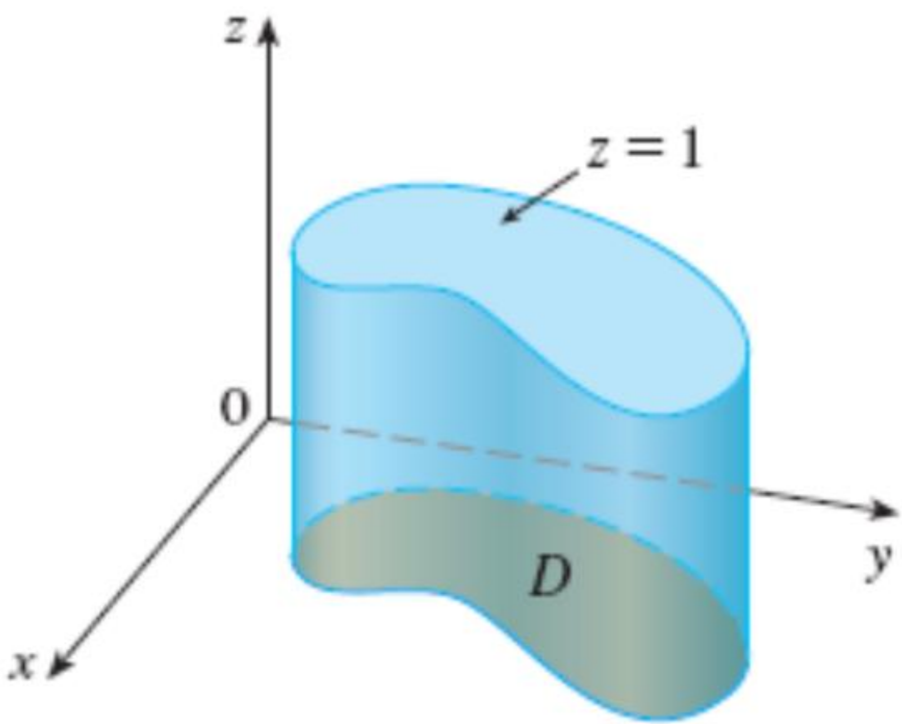
1. Double Integrals Over General Regions

Some properties of double integrals

10 $$\iint_D 1\, dA = A(D)$$ the area of region D



1. Double Integrals Over General Regions

Some properties of double integrals

If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

11
$$mA(D) \leqslant \iint_D f(x, y)\, dA \leqslant MA(D)$$

1. Double Integrals Over General Regions

Ex 6:

Use property 11 to estimate $\iint\limits_D e^{\sin x \cos y}\, dA$ where D is the disk whose center is the origin and whose radius is 2.